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ON 2-SPHERICAL CELL-LIKE 2-DIMENSIONAL PEANO CONTINUUM

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We report about joint with Katsuya Eda and Dušan Repovš result:

There exists 2-spherical simply connected cell-like 2-dimensional Peano continuum X.

First of all we fix the terminology. By n-*spherical space* we mean a space $n's$ homotopy group of which is nontrivial. The space is called *cell-like* if it has trivial shape. By *Peano contimuum* we mean compact connected locally connected metric space. By dimension we mean Lebesgue dimension.

The space X is constructed as follows. Consider the closed topologist's sine curve on the square $I^2 = [0,1] \times [-\frac{1}{2}, \frac{1}{2}] \subset \mathbb{R}^2$:

$$T = \left\{ (x,y) \in \mathbb{R}^2 |\ 0 < x \leq 1, y = \frac{1}{2}\sin\left(\frac{2\pi}{x}\right) \right\} \cup (\{0\} \times [-1,1]).$$

Let S^1 be the circle and s_0 be any of its points which we consider as base point. Consider the topological sum of I^2 and $T \times S^1$. The space X is the quotient space of this sum obtained by identification of the points (t, s_0) with $t \in T \subset I^2$ and by identification of each set $\{t\} \times S^1$ with t when $t \in 0 \times [-\frac{1}{2}, \frac{1}{2}] \subset I^2$.

Let G be any multiplicative group. By commutator $[x,y]$ of two elements x and y of group G we mean the element $xyx^{-1}y^{-1}$.

Commutator length $cl(g)$ of $g \in G$ is the minimal number n such that $g = \prod_{i=1}^{n}[x_i, y_i]$ [1, 4]. If such number does not exists then $cl(g) = \infty$. The commutator length $cl(g)$ is finite if and only if $g \in G'$ (G' is commutator subgroup of G). The terms *genus* for this concept is used in the literature [2].

Obviously, X is a cell-like Peano continuum. It was shown in [5] that this space is simply connected. Therefore it is necessary to show only that X is 2-spherical, i.e. there exists a nontrivial 2-dimensional singular cycle in X.

Let p be the natural projection of X onto I^2 which we consider as a subspace of the plane $\mathbb{R}^2$ with axis OX and OY. Let $I^2_+ = \{(x,y) \in I^2 | \ y \geq 0\}$, $I^2_- = \{(x,y) \in I^2 | \ y \leq 0\}$, $A^+ = p^{-1}(I^2_+)$, $A^- = p^{-1}(I^2_-)$.

Since the pair $\{A^+, A^-\}$ is an excisive couple of subsets we have the Mayer-Vietoris exact sequence ([10], p.188):

$$H_2(X) \xrightarrow{\delta} H_1(A^+ \cap A^-) \xrightarrow{(i_1,\ i_2)} H_1(A^+) \oplus H_1(A^-).$$

Obviously, the spaces $A^+ \cap A^-$, A^+ and A^- are homotopy equivalent to the Hawaiian earrings. To show that $H_2(X) \neq 0$ it suffices to prove that $i = (i_1,\ i_2)$ is not a monomorphism. Consider the natural circles $\{S^1_n\}_{n \in N}$ of the space $A^+ \cap A^-$ with the clockwise orientation (We consider $A^+ \cap A^-$ as a subspace of the plane XOZ). Let a_n be the element of $\pi_1(A^+ \cap A^-)$ corresponding to the loop winding once around the circle S^1_n in the positive direction.

Let a^+ be element of fundamental group $\pi_1(A^+ \cap A^-)$ generated by loop winding consecutively once around each circle $\{S^1_n\}_{i=1}^{\infty}$ in positive direction odd circles and in negative direction even circles. Element a^- is defined similar way but corresponding loop winds in negative direction all odd circles and in positive direction even circles. Schematically elements a^+ and a^- could be expressed as

$$a^+ = a_1 a_2^{-1} a_3 a_4^{-1} \cdots a_{2n-1} a_{2n}^{-1} \cdots$$

and

$$a^- = a_1^{-1} a_2 a_3^{-1} a_4 \cdots a_{2n-1}^{-1} a_{2n} \cdots .$$

Let $a \doteq a^+ a^-$. Since the 1-dimensional homology group is the abelanization of the fundamental group of the corresponding space, we have element $[a] \in H_1(A^+ \cap A^-)$.

Obviously, $a_1 = a_2, a_3 = a_4, \ldots, a_{2n-1} = a_{2n}, \ldots$ in $\pi_1(A^+)$ and $i_1([a]) = 0$.

Since $a_2 = a_3, a_4 = a_5, \ldots, a_{2n} = a_{2n+1}, \ldots$ in $\pi_1(A^-)$ we have $i_2[a] = [a_1^{-1} a_1] = 0$.

Therefore $i(a) = (i_1(a), i_2(a)) = 0$. So it is enough to show that $[a] \neq 0$ in $H_1(A^+ \cap A^-)$ or that a is not a element of commutator subgroup of $\pi_1(A^+ \cap A^-)$. Suppose that a lies in commutator subgroup, then $cl(a) = m$ for some number m. To prove that this is not possible we shall need some algebraic lemmas.

Lemma 0.1. *For any elements $\{b_i\}_{i=1}^n$ of any group G there exist elements $\{x_i\}_{i=1}^n$ of the group G such that:*

$$b_1 b_2 \cdots b_{2n} b_1^{-1} b_2^{-1} \cdots b_{2n}^{-1} = [x_1, x_2][x_3, x_4] \cdots [x_{2n-1}, x_{2n}].$$

If group G is free group and the set of elements $\{b_i\}_{i=1}^n$ is a basis of the group G then $\{x_i\}_{i=1}^n$ is also a basis of G.

Proof. It is easy to check by inductionthat the set of elements:
$x_1 = b_1,$
$x_2 = b_2,$
$x_3 = b_2 b_1 b_3,$
$x_4 = b_4 b_1^{-1} b_2^{-1},$
...
$x_{2n-1} = b_{2n-2} b_{2n-3} \cdots b_2 b_1 b_{2n-1},$
$x_{2n} = b_{2n} b_1^{-1} b_2^{-1} \cdots b_{2n-2}^{-1}$
satisfy the condition of the lemma. □

Choose a natural number n such that $n > m$. Consider the projection f of the group $\pi_1(A^+ \cap A^-)$ on the free group F_{2n} with $2n$ generators $b_1, b_2, \cdots, b_{2n}$, which is defined as follows $f(a_1) = b_1, f(a_2) = b_2^{-1}, \ldots, f(a_{2n-1}) = b_{2n-1}, f(a_{2n}) = b_{2n}^{-1}$, for $i > 2n, f(a_i) = e$, where e is the trivial element of F (Such projection is generated by continuous mapping of the space $A^+ \cap A^-$ to the first $2n$ circles). Then $f(a) = b_1 b_2 \cdots b_{2n} b_1^{-1} b_2^{-1} \cdots b_{2n}^{-1}$. Since f is a homomorphism and by our hypothesis $cl(a) = m$ it follows that $cl(f(a)) \leq m$. However, by Lemma 0.1

$$b_1 b_2 \cdots b_{2n} b_1^{-1} b_2^{-1} \cdots b_{2n}^{-1} = [x_1, x_2][x_3, x_4] \cdots [x_{2n-1}, x_{2n}]$$

and by the following proposition:

Proposition 0.2. ([9], p.55, [2], p.137). *If F is a free group with a basis of distinct elements $x_1, x_2, \ldots x_{2n}$ and there are elements $u_1, u_2, \ldots, u_{2m}$ of F such that*

$$[x_1, x_2][x_3, x_4] \cdots [x_{2n-1}, x_{2n}] = [u_1, u_2][u_3, u_4] \cdots [u_{2m-1}, u_{2m}]$$

then $m \geq n$.

it follows that $cl(f(a)) = n$. This contradicts our choice of number n. Therefore the element $[a]$ is a nontrivial element of $Ker(i)$ and $H_2(X) \neq 0$.

Since $\pi_1(X) = 0$, it follows by the by Hurewicz Theorem that $\pi_2 = H_2$ and $\pi_2(X) \neq 0$.

Problem 0.3. *Does there exists a noncontractible finite-dimensional Peano continuum all homotopy groups of which are trivial?*

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